NUMERICAL SOLUTION FOR THE POTENTIAL BETWEEN PARALLEL PLATES

I. INTRODUCTION

The electric scalar potential due to a surface charge distribution ρ_S is given by

$$V = \frac{1}{4\pi\varepsilon_o} \iint_S \frac{\rho_s}{\left|\vec{R} - \vec{R}'\right|} ds' \tag{1}$$

where $\vec{R}$ is the position vector to the observation point at which the potential is to be determined and $\vec{R}'$ is the position vector to a source point. For most problems ρ_S is unknown, but the potential at certain boundaries is known (the "boundary conditions"). An example is the parallel plate capacitor shown in Figure 1. The charge distribution on the plates is not known, but the potential difference between them is.

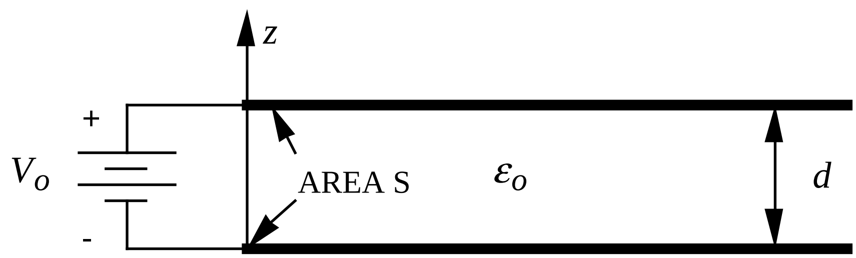


Figure 1. Parallel plate capacitor.

II. THE METHOD OF MOMENTS

Equation (1) can be solved numerically using a technique called the method of moments (MM). The unknown charge distribution is expanded into a series

$$\rho_s = \sum_{i=1}^{K} a_i \, p_i(x', y') \tag{2}$$

where $p_i(x', y')$ are the basis functions and a_i the expansion coefficients. The expansion functions are chosen to fit the problem. They can be sinusoids, delta functions, step functions, or exponentials to name a few. For this particular problem, two-dimensional pulses (or pedestals) are appropriate. The final charge distribution will be the two-dimensional equivalent of a step approximation as depicted in Figure 2.

The first step in the MM solution is to subdivide the plate into rectangular patches of dimension Δ_x by Δ_y as shown in Figure 2. The rectangles are called subdomains. They must be small enough so that the surface charge density on each rectangle is approximately constant. The patches are indexed from 1 to M in x and from 1 to N in y. The center points are given by the coordinates

$$
\begin{aligned}
x_{mn} &= [1/2 + (m-1)]\Delta_x \\
y_{mn} &= [1/2 + (n-1)]\Delta_y \\
z_{mn} &= \begin{cases} 0 & \text{bottom plate} \\ d & \text{top plate} \end{cases}
\end{aligned}
\tag{3}
$$

The procedure must be applied to both the top and bottom plates. Therefore the total number of patches is $K = 2MN$. If the patches are numbered consecutively (counting along x first, starting with the bottom plate) then

$$
i = m + (M-1)(n-1), \begin{cases} i = 1, \ldots, \frac{K}{2} & \text{for bottom plate} \\ i = \frac{K}{2}+1, \ldots, K & \text{for top plate} \end{cases}
\tag{4}
$$

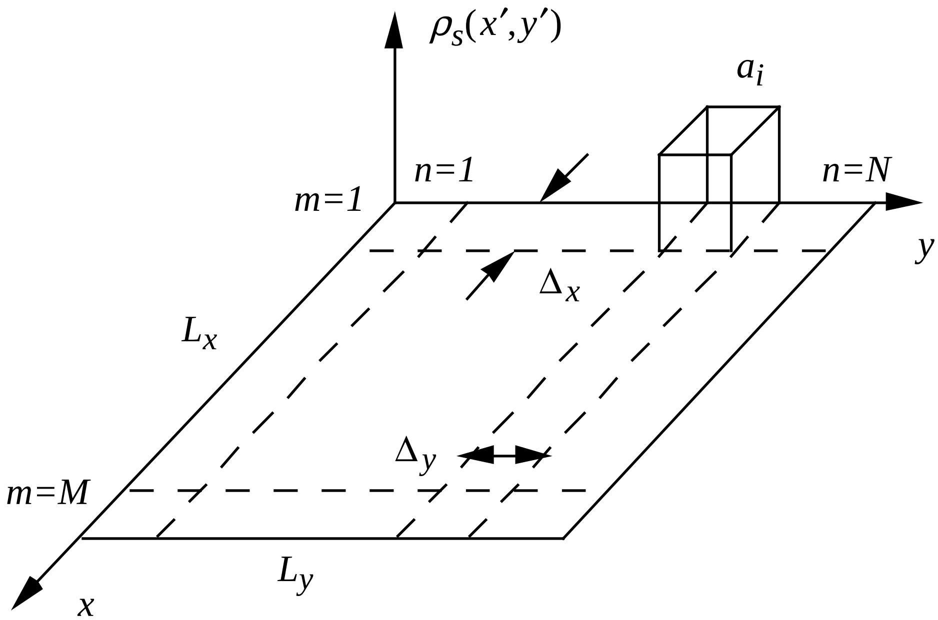


Figure 2: Discretizing the plate into rectangular subdomains.

The series coefficients are determined by substituting (2) into (1) and then applying the boundary conditions. The position vectors are

$$
\begin{aligned}
\vec{R} &= \hat{x}x + \hat{y}y + \hat{x}z \\
\vec{R}' &= \hat{x}x' + \hat{y}y' + \hat{z}z' \\
\left|\vec{R} - \vec{R}'\right| &= \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}
\end{aligned}
\tag{5}
$$

For the bottom plate $z' = 0$ and for the top plate $z' = d$. Inserting (2) into (1) gives

$$V = \frac{1}{4\pi\varepsilon_o} \int_0^{L_x} \int_0^{L_y} \underbrace{\left(\sum_{i=1}^{K} a_i \, p_i(x', y') \right)}_{\rho_s} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z)^2}} dx' dy' \tag{6}$$

The basis function $p_i(x', y')$ is only nonzero on patch number i. Therefore the integration and summation can be interchanged and the limits of integration changed to those of patch i

$$V = \frac{1}{4\pi\varepsilon_o} \sum_{i=1}^{K} a_i \left(\int_{x_i - \frac{\Delta x}{2}}^{x_i + \frac{\Delta x}{2}} \int_{y_i - \frac{\Delta y}{2}}^{y_i + \frac{\Delta y}{2}} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z)^2}} dx' dy' \right) \tag{7}$$

The integrals can be evaluated numerically. Although the charge is assumed to be constant on each patch, because of the square root term in the denominator, the integrand is not constant at all points on the patch. However, we shall assume that the integrand changes slowly enough so that the situation in Figure 3 holds. That is, the value of the integrand at all points on the plate is approximately equal to the value at the center. In this case

$$\int_{x_i - \frac{\Delta_x}{2}}^{x_i + \frac{\Delta_x}{2}} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z)^2}} dx' \approx \frac{\Delta_x}{\sqrt{(x-x_i)^2 + (y-y)^2 + (z)^2}} \tag{8}$$

and for the y integration

$$\int_{y_i - \frac{\Delta_y}{2}}^{y_i + \frac{\Delta_y}{2}} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z)^2}} dy' \approx \frac{\Delta_y}{\sqrt{(x-x)^2 + (y-y_i)^2 + (z)^2}} \tag{9}$$

Finally, combining (7), (8) and (9)

$$V \approx \frac{1}{4\pi\varepsilon_o} \sum_{i=1}^{K} \frac{a_i \Delta_x \Delta_y}{\sqrt{(x-x_i)^2 + (y-y_i)^2 + (z)^2}} \tag{10}$$

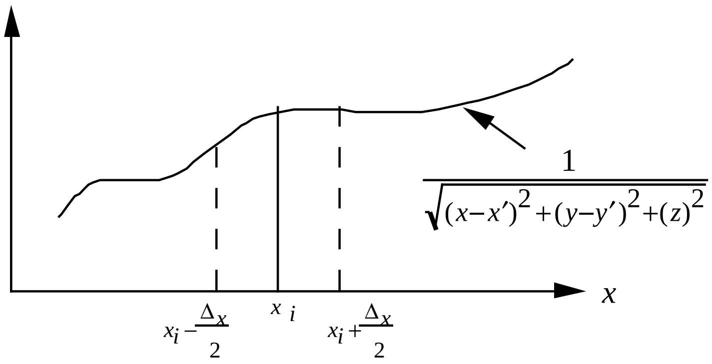


Figure 3: Approximation for the integration in x'.

III. BOUNDARY CONDITIONS

To determine the expansion coefficients the boundary conditions are enforced at the center of each patch. There are $K/2$ equations for the bottom plate of the form

$$V(x_j, y_j, 0) = 0 \approx \frac{\Delta_x \Delta_y}{4\pi\varepsilon_o} \sum_{i=1}^{K} \frac{a_i}{\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}} \qquad j = 1, 2, \ldots, \frac{K}{2} \tag{11}$$

and similarly on the top plate

$$V(x_j, y_j, d) = V_o \approx \frac{\Delta_x \Delta_y}{4\pi\varepsilon_o} \sum_{i=1}^{K} \frac{a_i}{\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}} \qquad j = \frac{K}{2} + 1, \frac{K}{2} + 2, \ldots, K \tag{12}$$

where V_o is the known voltage of the battery. Equations (11) and (12) can be cast into matrix form

$$[V] = [Z][A] \tag{13}$$

where,

$$[V] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \cdots \\ V_o \\ \vdots \\ V_o \end{bmatrix} \qquad [A] = \begin{bmatrix} a_1 \\ \vdots \\ a_{K/2} \\ \cdots \\ a_{K/2+1} \\ \vdots \\ a_K \end{bmatrix} \tag{14}$$

$$[Z] = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1K} \\ Z_{21} & Z_{22} & & \\ \vdots & & \ddots & \vdots \\ Z_{K1} & \cdots & & Z_{KK} \end{bmatrix} \tag{15}$$

$$Z_{ij} = \frac{\Delta_x \Delta_y}{4\pi\varepsilon_o} \sum_{i=1}^{K} \frac{a_i}{\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}} \qquad j = 1,2,\ldots,K \tag{16}$$

Note that the matrix Z is symmetric; that is $Z_{ij} = Z_{ji}$. The elements of Z account for interactions between the charges on all patches, not only the same plate but on the neighboring plate as well. The expansion coefficients are obtained by solving the matrix equation

$$[A] = [Z]^{-1}[V] \tag{17}$$

IV. SINGULARITIES

Problems occur when $i = j$ because the denominator in (16) becomes zero. This case must be treated separately. To avoid the singularity note that Z_{ij} represents the potential at the center of patch i due to the charge on patch j. We can calculate this by approximating the rectangular patch by a circular one with an equivalent radius that gives the same area as the rectangular patch

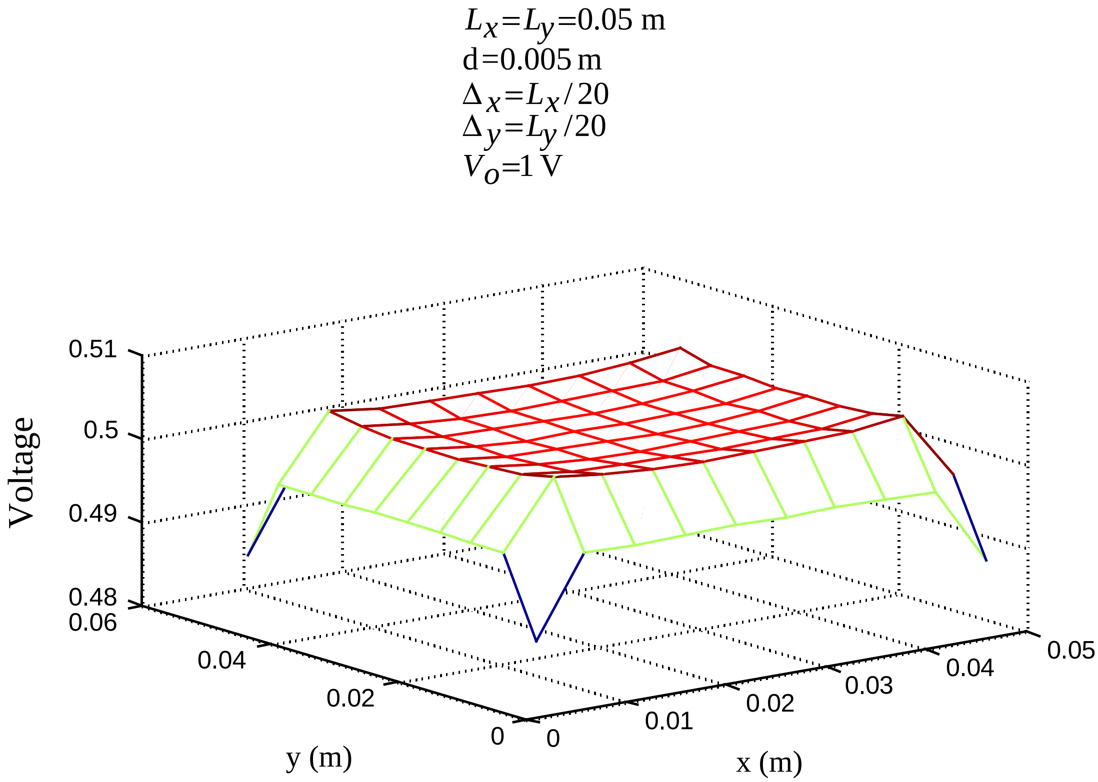
$$\pi b^2 \approx \Delta_x \Delta_y \quad \Rightarrow \quad b \approx \sqrt{\frac{\Delta_x \Delta_y}{\pi}} \tag{18}$$

Chang has solved the circular disk problem (Example 3-8). Using $z = 0$ in equation 3-41 (with $\rho_s = 1$) gives an approximation for the "self term"

$$Z_{ij} \approx \frac{b}{2\varepsilon_o} = \frac{1}{2\varepsilon_o}\sqrt{\frac{\Delta_x \Delta_y}{\pi}} \tag{19}$$

V. EXERCISES

Calculate the potential distribution on a surface midway between the top and bottom plates, $z = d/2$. Program your solution in Matlab, Mathcad, Mathematica or other high level language. Use the following values in your calculation (result shown below):



$L_x = L_y = 0.05$ m
d=0.005 m
$\Delta_x = L_x/20$
$\Delta_y = L_y/20$
$V_o = 1$ V
Voltage
0.51
0.5
0.49
0.48
0.06
0.04
0.02
0
y (m)
0
0.01
0.02
0.03
0.04
0.05
x (m)